Freeze-out Mergers

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Abstract

Do freeze-out mergers mitigate the free rider problem of corporate takeovers? We study this question in a tender offer model with finitely many shareholders. Under a freeze-out merger, shareholders expect to receive the original offer price whether or not they tender their shares. We show that the ability to freeze-out minority shareholders increases the raider's expected profit, and this profit is higher when the ownership requirement the acquirer has to meet in order to complete a freeze-out merger is lower. Furthermore, the raider's expected profit decreases as the firm becomes more widely held. However, in the limit, for any ownership requirement that is more stringent than simple majority, the raider's expected profit converges to zero. In this sense, freeze-out mergers do not provide a solution to the free rider problem.

Keywords: Free rider problem, Freeze-out merger, Takeovers, Tender offer.

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1 Introduction

The ability of the market for corporate control to efficiently allocate resources is much debated. A seminal paper by Grossman and Hart (1980) argued that there is a free-rider problem that prevents acquirers from successfully taking over companies that are widely held. The crucial assumption in this argument is that target shareholders do not view themselves as pivotal in the success of the takeover. Therefore, each shareholder refuses to tender his share whenever he expects the post takeover value to be higher than what is being offered. If all shareholders behave that way then a value increasing acquirer cannot convince shareholders to tender their shares and at the same time make a profit on the purchased shares. Without any private benefits of control, free-riding precludes efficient takeovers, leading to an inefficient market for corporate control.

As a remedy to the free-rider problem, Grossman and Hart argued in favor of diluting non-tendering shareholders, thereby excluding them from any gains due to the takeover. Subsequently, the academic literature has identified several mechanisms that could potentially mitigate the free-rider problem,[1] most notably, freeze-out mergers. A freeze-out merger is a transaction in which the controlling shareholder buys out the shares of the minority, delists the corporation, and then takes it private. The rules that govern freeze-out mergers vary considerably across jurisdictions, but they all limit the ability of shareholders to retain their shares in the target company as a minority after the acquirer gained control. In some jurisdictions, the acquirer is allowed to freeze-out shareholders only after gaining a supermajority of the votes (for example, the freeze-out threshold is two thirds in Texas and New York, and as high as 90% in Canada and most European countries), whereas a simple majority is sufficient in Delaware and most other U.S. states. In many instances, the price at which the acquirer can freeze-out minority shares is equal to the original tender offer price, although this rule also varies from

[1] See, for example, Shleifer and Vishny (1986), Hirshleifer and Titman (1990), Kyle and Vila (1991), Burkart, Gromb, and Panunzi (1998), Cornelli and Li (2002). Müller and Panunzi (2004), At, Burkart, and Lee (2011), Marquez and Yilmaz (2012).

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one jurisdiction to another.[2,3,4,5]

How can a freeze-out merger solve the free-rider problem? The idea is as follows. If shareholders believe that the freeze-out threshold will be met (i.e., the fraction of tendered shares will exceed the ownership fraction that is required for freezing out minority shareholders), then each shareholder is indifferent between tendering and retaining his shares, irrespective of the offer price. Therefore, there always exists an equilibrium in which all shareholders tender even at a price significantly lower than the post takeover value. In this equilibrium, a value-increasing acquirer ("raider") can profitably take over the target firm, thereby restoring economic efficiency.[6] If shareholders are infinitesimal, this argument holds for any freeze-out threshold strictly smaller than 100%. Building on this line of arguments, it has been argued that freeze-out mergers are the perfect panacea for the free-rider problem (e.g., Amihud, Kahan and Sundaram (2004)).

In this paper, we challenge this proposition. We show that unless the most extreme freeze-out clause is considered (i.e., a 50% freeze-out threshold; the simple majority), freeze-out mergers do not solve the free-rider problem as long as shareholders can be pivotal for the takeover, even if the probability of being pivotal is arbitrarily small. Specifically, following Bagnoli and Lipman (1988), we analyze a tender offer model with finitely many shareholders. Such a model allows each individual shareholder to have an impact on the success of the tender offer. In this model, there exists an equilibrium in which each shareholder is more likely to tender when the offer price is higher.[7] Generally, this equilibrium does not exist

[2]See Ventoruzzo (2010) for a detailed discussion of differences on regulation of freeze-out transactions between U.S. and Europe.

[3]In most U.S. states, including Delaware, if the acquirer owns more than 90% of shares he can buy the remaining minority shares without a shareholder vote through a short-form merger (Gilson and Black, 1995).

[4]U.S. corporations can also impose supermajority requirements through their charters (see Gomes (2001) and Amihud, Kahan and Sundaram (2004)). Acquierers also tend to include a nonwaivable majority-of-the-minority closing condition to obtain a favorable treatment from the courts (Subramanian (2007)). These provisions effectively implement a supermajority freeze-out threshold.

[5]In Canada, the acquirer can freeze-out minority shareholders without a shareholder vote as long as he acquired at least 90% of the shares at a tender offer. A two third majority may be sufficient as long as the buyer satisfies additional requirements under Multilateral Instrument 61-101.

[6]If the offer is conditional on obtaining majority of the votes, there also exists an equilibrium in which the offer fails and the freeze-out clause has no impact. See Section 4 for our analysis of conditional offers.

[7]With freeze-out mergers, there also exist equilibria in which shareholders tender irrespective of the offer price. In Section 3.3 we show that these equilibria are not robust.

in models with infinitesimal shareholders. Bagnoli and Lipman (1988) showed that although the raider can have a strictly positive expected profit in equilibrium, this profit converges to zero as the number shareholders gets arbitrarily large, in line with Grossman and Hart predictions. Drawing from the existing literature, one might conclude that this argument is invalid when freeze-out mergers are considered. To examine this proposition, we add freeze-out mergers to their setup. As one might expect, the ability to freeze-out shareholders increases the raider's expected profit, and this profit is higher for lower (i.e., stricter) freeze-out thresholds. Furthermore, the expected profit decreases as the firm becomes more widely held. However, our main result shows that as the number of shareholders gets arbitrarily large, the raider's expected profit in equilibrium converges to zero for *any* freeze-out clause with an ownership threshold that is strictly above simple majority. That is, freeze-out mergers do not solve the free-rider problem in widely held firms.[8]

The intuition behind our result is the following. As long as the freeze-out threshold is strictly greater than 50%, shareholders can never rule out in equilibrium the possibility that the takeover will succeed but the number of tendered shares will be insufficient to trigger the freeze-out clause. In those events, minority shareholders receive the post takeover value. Therefore, conditional on the success of the takeover, non-tendering shareholders can expect to get on average strictly more than the offer price. If the takeover is expected to succeed as the number of shareholders gets arbitrarily large, each shareholder believes that the impact of his individual decision is negligible, and the incentives to free-ride prevail just as in the Grossman and Hart (1980) model.

This argument, however, is incomplete. In equilibrium, shareholders are more likely to tender their shares when the offer price is higher. Therefore, if the price is sufficiently high but still below the post takeover value, the probability that the number of tendered shares exceeds the freeze-out threshold must converge to one as the number of shareholders gets arbitrarily large. If so, shareholders should have no incentive to free-ride. Without free-riding, the raider should be able to take over the firm and make a strictly positive profit. But this argument

[8]Among other things, our theory suggests that the effect of freeze-out mergers on the free-rider problem (and therefore, on the market for corporate control) crucially depends on the employed freeze-out threshold, which is an intriguing empirical question.

is false. To understand why, note that shareholders are always *indifferent* between tendering and not tendering whenever the freeze-out clause is triggered; they will get the same offer price either way. Because of their indifference, each shareholder *conditions* his decision to tender only on events in which the freeze-out clause is *not* triggered. As was explained above, conditional on those events, shareholders are strictly better off not tendering their shares. Therefore, even if shareholders were to believe that the freeze-out clause will almost surely be triggered, their incentives to free-ride would not be affected. The raider can induce them to tender their shares only if the offer price converges to the post takeover value, which would leave the raider with no profit in the limit. The free-rider problem remains unsolved.

Our paper is related to Bagnoli and Lipman (1988), Holmstrom and Nalebuff (1992), Gromb (1993), Cornelli and Li (2002), Marquez and Yılmaz (2008), Dalkır and Dalkır (2014), Dalkır (2015), and Ekmekci and Kos (2016), who also studied tender offer models with a finite number of shareholders. None of these papers, however, studies the effect of freeze-out mergers on the free-rider problem. Freeze-out mergers were studied by Yarrow (1985), Burkart, Gromb, and Panunzi (1998), Gomes (2001), Amihud, Kahan and Sundaram (2004) and Maug (2006), who argue that freeze-out mergers can mitigate the free-rider problem.[9] Our analysis highlights that this conclusion is sensitive to the assumption that shareholders are infinitesimal and to the assumption that the freeze-out threshold is exactly the majority requirement.[10] One exception to this literature is Müller and Panunzi (2004), who claim that under the existing legal framework in the U.S., there always exists a risk that the raider would fail to freeze out target shareholders even if the number of tendered shares is sufficient to trigger the freeze-out clause (e.g., because the raider has violated his fiduciary duty). Since under this assumption the expected consideration to the non-tendering shareholders is strictly higher than the original tender offer price, these authors argue that in a model with infinitesimal shareholders, freeze-out mergers are completely ineffective in solving the free-rider problem. By contrast, in our

[9] Interestingly, Burkart and Lee (2018) argue that in a tender offer model with post-takeover moral hazard the option to freeze-out can exacerbate the free-rider problem.

[10] If the freeze-out threshold is exactly simple majority (or if the offer requires the number of tendered shares to exceed the freeze-out threshold) shareholders can never free-ride since by design the tender offer cannot succeed without triggering the freeze-out clause. However, in Section 4.1, we show that this result is not robust to small perturbations of the model, and in general, the free-rider problem is not solved even in those special cases.

model the risk of a freeze-out failure is not deus ex machina, but rather endogenously created by the mixed tendering strategies of the (finite number of) target shareholders in equilibrium. Therefore, unlike the analysis in Müller and Panunzi (2004), our results are robust to changes of the regulatory and institutional framework that would eliminate the legal risk that a freeze-out merger would fail.[11] Moreover, since in our framework the risk of a failed freeze-out merger is endogenous, our analysis provides novel predictions on the effect of freeze-outs on the success probability of a takeover, the expected takeover premium, and the raider's profit.

2 The Model

There are n shareholders indexed by i, each owns a single share in the firm, and each share carries one vote. In addition, there is a raider, who wishes to purchase and run the firm, and an incumbent, who currently manages the firm. We normalize the per-share value under the incumbent management to zero. The raider, if successful in taking over the firm, expects to create a firm value of $V > 0$.[12] The per-share value can therefore be stated as $v = \frac{V}{n}$. The raider may also have private benefit of control, $B \geq 0$, which she receives if and only if the takeover succeeds.

The raider makes a tender offer at price $P \in [0, \infty)$ or $p = \frac{P}{n}$ per share. We focus on unconditional offers, i.e., the raider's offer is valid for any and all shares. Conditional offers are considered in Section 4. All shareholders decide simultaneously whether to tender or to reject the offer. The set of actions for shareholder i is defined by $\{tender, keep\}$. A (mixed) strategy of shareholder i in a tender offer subgame is denoted by $\sigma_i \in [0, 1]$, where σ_i is the probability that shareholder i tenders. If the number of shares tendered, n_T, is equal to or exceeds those that are kept, then raider takes over. Otherwise, incumbent stays in office. That is, the raider needs to purchase at least half of the outstanding shares for a successful takeover.[13]

[11] In addition, in Section 3.3 we show that our results are robust to various perturbations of the model, including shareholder inattention and tendering costs.

[12] It can be shown that if the raider decreases the value of the target post-takeover (i.e., $V < 0$), then the introduction of a freeze-out clause does not change the analysis in any material way. In particular, without additional provisions, freeze-out mergers are also ineffective in preventing value destroying takeovers.

[13] Similar results hold with supermajority rules, as long as the freeze-out threshold ϕ is strictly higher than the required supermajority threshold.

The freeze-out threshold is denoted by $\phi \in (\frac{1}{2}, 1]$. If the number of tendered shares is greater than $n\phi$, then all shares are bought at price p. In other words, shareholders cannot free ride if sufficiently large fraction of other shareholders tender. Note that Bagnoli and Lipman's (1988) model is a special case of our model when $\phi = 1$. In Section 4.1 we also discuss the knife edge case $\phi = \frac{1}{2}$.

3 Analysis

We analyze the symmetric subgame perfect Nash equilibrium of this game. All omitted proofs are in the Appendix.

3.1 Tendering subgame

We start with the characterization of the symmetric Nash equilibrium of the tendering subgame. Note that offering a price strictly above v (below 0) per share is suboptimal, as all shareholders accept (reject) such offer for sure. Without loss of generality we restrict attention to $p \in [0, v]$.

When the offer is unconditional, there are two types of symmetric Nash equilibrium of the tendering subgame. In one type of equilibria, shareholders do not entertain the possibility of affecting the outcome, and the tendering decision does not depend on the offer price. In these equilibria, which exist for any non-negative offer, all shareholders tender their shares with probability one. Alternatively, there is a second type of equilibria in which shareholders do not rule out the possibility of being pivotal. In these equilibria the tendering decision depends on the offer price. We call this type of equilibrium, a *responsive* equilibrium (and thus, the other type is called a non-responsive equilibrium). We will characterize the unique responsive symmetric equilibrium. In Section 3.3, we motivate the selection of the responsive equilibrium by its empirical relevance and robustness to various perturbations of the model.

Consider the incentives of a shareholder to tender his share in a symmetric equilibrium. Since the offer is unconditional, if the shareholder tenders his share then he receives p regardless of the decision of other shareholders. If the shareholder keeps his share, he gets

$$K(\sigma,p)=v\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}+p\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}. \tag{1}$$

The first term in $K(\sigma,p)$ is the expected value of a share if the takeover succeeds but the freeze-out clause is not triggered. This event takes place if the number of tendered shares among all other $n-1$ shareholders is greater than or equal to $\frac{n}{2}$ but smaller than $n\phi$. The ability of a non-tendering shareholder to enjoy the full post takeover value of the firm if the takeover succeeds is the key behind the free-riding argument of Grossman and Hart (1980). The second term in $K(\sigma,p)$ is the expected value of a share if the freeze-out clause is triggered, that is, if the number of tendered shares is greater than or equal to $n\phi$. In this case, the non-tendering shareholder is forced to tender his share for p.

In equilibrium, each shareholder tenders (keeps) his share only if p is greater (smaller) than $K(\sigma,p)$, and is indifferent if $p=K(\sigma,p)$. A symmetric responsive equilibrium generally involves mixed strategies. Consequently, the equilibrium is characterized by $p=K(\sigma,p)$.

Proposition 1 (Tendering subgame) *For any $p\in[0,v]$ there exists a unique symmetric responsive equilibrium of the tendering subgame. In this equilibrium, each shareholder tenders his share with probability $\sigma(p)$, which is given by the unique solution of*

$$\frac{p}{v}=\frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}. \tag{2}$$

Moreover, $\sigma(p)$ increases in p and decreases in v and ϕ.

Proposition 1 shows that the unique responsive equilibrium has an interesting feature that tendering decisions depend on the offer price as characterized by (2), which is equivalent to $p=K(\sigma,p)$. The right hand side of (2) has the intuitive interpretation of being the probability that the takeover succeeds conditional on not triggering the freeze-out clause, from the perspective of a non-tendering shareholder. Naturally, this conditional probability increases with σ, the probability that each shareholder tenders his shares. Since the left hand side of

(2) increases with the offer price, the probability that shareholders tender their shares in equilibrium increases with p, as one might expect. Generally, since the tendering probability of each shareholder in the subgame increases in p, the raider will trade off a higher probability of success with a higher amount paid to shareholders.

The comparative statics of $\sigma(p)$ with respect to v and ϕ highlights the free-rider problem. Everything else being equal, higher v increases the benefit of each shareholder from holding onto his share if sufficiently many other shareholders tender their shares. Similarly, higher ϕ reduces the chances that the raider will freeze-out non-tendering shareholders, and hence, the benefit from not tendering (i.e., free-riding) increases.

3.2 Raider's Offer

In the previous subsection, we characterized the shareholders' tendering decisions for a given price $p \in [0, v]$. Next, using backward induction, we solve for the raider's offer price. In equilibrium, the raider offers $p^* \in \arg\max_{p \in [0,v]} \Pi(\sigma(p), p)$, where $\Pi(\sigma(p), p)$ is the expected profit for the raider given the equilibrium behavior in the tendering subgame. For a given σ and p,

$$\begin{aligned} \Pi(\sigma, p) &= v \sum_{j=\frac{n}{2}}^{n\phi-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} j - p \sum_{j=0}^{n\phi-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} j \\ &\quad + (v-p)\, n \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} + B \sum_{j=\frac{n}{2}}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j}. \end{aligned} \tag{3}$$

The first term in (3) is the raider's expected value of all tendered shares if the tender offer succeeds but the freeze-out clause was not triggered. The second term is the expected cost of the tender offer if the freeze-out clause was not triggered. Note that the raider has committed to buy all tendered shares, even if their number is smaller than $\frac{n}{2}$, the minimal number that is required for a successful takeover. The third term is the raider's expected profit times the probability that the freeze-out clause is triggered. Note that in this event the raider buys all shares, including those that were not tendered, for a price p. The last term is the raider's private benefit times the probability that the takeover succeeds.

Before we state our main result, it is important to emphasize that for any finite number of shareholders, freeze-out mergers increase the raider's expected profit.

Proposition 2 *The raider's expected profit,* $\Pi(\sigma(p), p)$*, decreases in* ϕ*.*

The intuition behind Proposition 2 follows from the observation that $\sigma(p)$ decreases in ϕ. A more aggressive freeze-out clause reduces the expected benefit of each shareholder from not tendering his share. Consequently, a lower ϕ allows the raider to sustain the same tendering probability with a lower offer price, which increases the raider's expected profit.

Next, we study the implications of freeze-out mergers in widely held firms, that is, when the number of shareholders is arbitrarily large. Our main result shows that in the absence of any private benefits of control, the raider's expected profit converges to zero for any $\phi \in (\frac{1}{2}, 1]$.

Proposition 3 *If* $B = 0$ *and the firm is widely held then the raider's expected profit is zero, that is,* $\lim_{n\to\infty} \Pi(\sigma(p), p) = 0$*.*

Why does the raider's expected profit converge to zero in spite of the freeze-out clause? The intuition is as follows. When the number of shareholders is large, by the law of large numbers, the fraction of tendered shares converges to the probability that each shareholder tenders his share. Let σ^*_∞ be the limit probability that a shareholder tenders his share under the unique symmetric responsive equilibrium. Similarly, define p^*_∞ as the corresponding limit offer. In the limit, the takeover fails unless $\sigma^*_\infty \geq \frac{1}{2}$. Since the raider can make a positive profit only if the takeover succeeds, we can restrict attention to $\sigma^*_\infty \geq \frac{1}{2}$.

There are two cases to consider. First, suppose $\sigma^*_\infty \in [\frac{1}{2}, \phi)$. Shareholders expect the takeover to succeed with a positive probability, but at the same time, they expect the number of tendered shares to be insufficient for the freeze-out merger. In fact, as the number of shareholders increases, they become more and more confident that the raider will not be able to freeze them out. In the limit, shareholders effectively ignore the possibility of a freeze-out, and as a result, the intuition is the same as in Bagnoli and Lipman (1988). Specifically, if $\sigma^*_\infty \in (\frac{1}{2}, \phi)$ then shareholders expect the takeover to succeed with probability one irrespective of their individual decision, and therefore, they have strong incentives to free-ride. The only way the raider can convince the shareholders to tender their shares is by offering them a price which is arbitrarily close to the post takeover value, v. In other words, $\sigma^*_\infty \in (\frac{1}{2}, \phi)$ requires $p^*_\infty = v$, which leaves the raider with a zero expected profit. By contrast, $p^*_\infty \in (0, v)$ requires $\sigma^*_\infty = \frac{1}{2}$, which implies that the success of the takeover is uncertain in the limit. As in Bagnoli and Lipman (1988), the raider's expected profit when the takeover succeeds is cancelled out by his losses when the takeover fails.

Second, suppose $\sigma^*_\infty \geq \phi$. The intuition here is more subtle. Notice that if $\sigma^*_\infty \geq \phi$ then shareholders not only expect the takeover to succeed as their number increases, but they also expect the raider to freeze them out. Seemingly, the benefit from free-riding should disappear in the limit, but in fact the opposite is true. To understand why, note that the decision to tender is inconsequential from the shareholder's perspective if the number of tendered shares exceeds the freeze-out threshold; either way, the shareholder receives p^*_∞. Therefore, shareholders ignore these events when deciding whether to tender their shares. The fact that these events become more likely when the number of shareholders increases (since $\sigma^*_\infty \geq \phi$) is immaterial for the argument; each shareholder conditions his decision to tender only on events in which the freeze-out threshold *is not met*. Conditional on these events, each shareholder faces the following trade-off from not tendering: He loses p^*_∞ if the takeover fails and gains v if the takeover succeeds. Therefore, shareholders are less likely to tender their shares if *conditional* on not meeting the freeze-out threshold the probability that the takeover succeeds is high. When $\sigma^*_\infty \geq \phi > \frac{1}{2}$, this conditional probability converges to one as the number of shareholders increases. In other words, shareholders become confident that the takeover will succeed, and therefore, they have strict incentives to free ride. Similar to the first case, the raider has to offer the post takeover value in order to convince shareholders to tender with a probability higher than ϕ. Therefore, reaching to an ownership level that allows the raider to freeze-out minority shareholders requires surrendering the entire surplus to shareholders, that is, $p^*_\infty = v$.

Proposition 4 *Consider the unique symmetric responsive equilibrium when the firm is widely held. Then,*

i.

$$\sigma^*_\infty = \min\left\{\frac{1}{2} + \frac{1}{2}\frac{B}{B+V}, \phi\right\}. \quad (4)$$

ii. *If $B = 0$ then $p^*_\infty \in (0, v)$ and the probability of a successful takeover is bounded away from one.*

iii. *If $B > 0$ then $p^*_\infty = v$ and,*

iv. *If $B > 0$ then the probability of a successful takeover is one.*

Proposition 4 characterizes the equilibrium in the limit. According to Proposition 3, the raider cannot make any profit on the shares he is buying; the only source of profit is his private benefits of control. Therefore, if $B > 0$ then the raider maximizes the probability that the takeover succeeds. The raider can guarantee the success of the takeover by offering shareholders $p^*_\infty = v$. Notice, however, that the optimal offer induces a tendering probability that is lower than ϕ, the freeze-out threshold. Intuitively, the raider cannot benefit from further increasing the price when the freeze-out clause is already expected to be triggered. Therefore, different from Bagnoli and Lipman (1988), $\sigma^*_\infty \leq \phi$ irrespective of B.

Proposition 4 also shows that if $B = 0$ then the optimal offer in the limit induces $\sigma^*_\infty = \frac{1}{2}$, the same as the solution of Bagnoli and Lipman (1988). Bagnoli and Lipman (1988) proved that the optimal offer maximizes the probability that each shareholder is pivotal for the success of the takeover, which is obtained by setting $\sigma = \frac{1}{2}$. However, their result holds for any finite n, not just in the limit when the number of shareholders is large. By contrast, with freeze-out mergers, the raider has additional incentives to increase the offer. Indeed, a higher offer increases the probability that the freeze-out clause is triggered, which benefits the raider. If ϕ is sufficiently close to 50% (from above), the benefit from increasing the offer is particularly high since triggering the freeze-out clause is easier. Indeed, we show in the Appendix that for a finite n, the optimal offer induces $\sigma > \frac{1}{2}$ if ϕ is sufficiently close to 50%. However, as we explained in the discussion that follows Proposition 3, as the number of shareholders gets larger, sustaining a tendering probability strictly higher than $\frac{1}{2}$, which is necessary for triggering the freeze-out clause, becomes too costly for the raider when $B = 0$. Therefore, in the limit, the tendering probability is the same as if there is not freeze-out. This result holds for any $\phi > \frac{1}{2}$.

Overall, Proposition 3 and Proposition 4 show that the limit of the responsive equilibrium in a model with a finite number of shareholders is fundamentally different from the equilibrium in a game with infinitesimal shareholders. Intuitively, as long as there is a strictly positive probability that the takeover succeeds but the freeze-out threshold is not met, shareholders have strict incentives to free ride even if the probability of this event is arbitrarily small. If shareholders are infinitesimal this probability is assumed to be exactly zero. Arguably, under this assumption, freeze-out mergers solve the free-rider problem. The contrast with our result suggests that this argument is fragile not only to the assumption that the freeze-out condition holds with complete certainty, as argued by Müller and Panunzi (2004), but also to the assumption that shareholders are infinitesimal. In fact, this intuition is also related to the fragility of the non-responsive equilibrium in our model, as we discuss in detail in Section 3.3 below.

3.3 Discussion of equilibrium selection

Our analysis focuses on the unique symmetric responsive equilibrium. However, with freeze-outs, for any non-negative offer there also exists an equilibrium in which all shareholders tender their shares with probability one. In this non-responsive equilibrium, free-riding has no effect, and the outcome Pareto dominates the symmetric responsive equilibrium that we analyzed. Arguably, since this non-responsive equilibrium exists with a freeze-out merger but not otherwise, the introduction of freeze-out mergers eliminates the free-rider problem. Therefore, our argument that freeze-out mergers do not solve the free-rider problem in tender offers (when ownership is widely dispersed) requires a justification for the selection of the responsive equilibrium.

We justify our selection on two grounds. First, unlike non-responsive equilibria, the symmetric responsive equilibrium has properties that are consistent with the empirical evidence. Indeed, in a non-responsive equilibrium the takeover premium is zero (or arbitrarily small) and the offer succeeds with probability one. In practice, a nontrivial number of takeovers do fail, and a zero takeover premium is inconsistent with the well documented evidence that target shareholders obtain most of the surplus from M&A transactions, especially so in tender offers.[14] By contrast, in a responsive equilibrium the surplus is divided between the raider and target shareholders (i.e., a strictly positive takeover premium, bounded away from zero), and the share of target shareholders increases as the ownership of the firm becomes more dispersed. Moreover, in a responsive equilibrium the takeover fails on the path with a strictly positive probability. In practice, there could be other factors that affect these empirical patterns, but in the spirit of Occam's razor, focusing on the responsive equilibrium seems as the natural and relevant selection.

Second, and more fundamentally, the symmetric responsive equilibrium is more robust than any non-responsive equilibrium. Specifically, below we consider three different (exogenous) perturbations of the model and show that while the responsive equilibrium survives these perturbations, the other equilibrium does not. Moreover, the equilibrium under these perturbations converges to the responsive equilibrium in our model. The fragility of the non-responsive equilibrium is not a coincidence: The non-responsive equilibrium builds on the indifference of shareholders between tendering and keeping their shares when they expect *all other shareholders to tender their shares with probability one.* Importantly, it does not only require shareholders to believe that the takeover will succeed with probability one, but it also requires the belief that all shareholders would tender their shares with probability one, which is a stronger requirement. This requirement is unlikely to hold in practice for a variety of reasons, as the three perturbations below demonstrate.[15] The robustness of the responsive equilibrium is also not a coincidence: A responsive equilibrium inherently allows shareholders' behavior to be perturbed by considering mixed strategies, and therefore, it can sustain additional perturbations of the model. In this respect, our analysis shows that the freeze-out rule being ineffective against free-riding in widely held firms is the robust equilibrium prediction.

[14] For a broad survey of the large empirical literature, see Bruner (2004).

[15] The fragility of the non-responsive equilibrium to perturbations of our model is analogous to the fragility of equilibria in which *conditional* tender offers fail with probability one (see Gromb (1993) and our analysis of conditional offers in Section 4). In both cases small perturbations of the model "break" the indifference of shareholders between tendering and not tendering.

3.3.1 Uncertainty about the freeze-out condition

In the spirit of Müller and Panunzi (2004), consider a perturbation of the model in which there is a small (legal) uncertainty about the freeze-out condition. For example, suppose that if more than ϕn shares are tendered, each non-tendering shareholder receives p with probability $1-\varepsilon$ and v otherwise, where $\varepsilon \in (0,1)$ can be arbitrary small. One might argue that with a finite number of shareholders the uncertainty on the freeze-out condition must be sufficiently large to make freeze-outs ineffective, but that is not the case. As long as $\varepsilon > 0$, a non-responsive equilibrium does not exist: If each shareholder expects all other shareholders to tender their shares, non-tendering yields $p + \varepsilon (v - p) > p$, which is strictly more than what a shareholder can get for his share by tendering.[16] At the same time, it can be shown that a unique symmetric responsive equilibrium always exists, and its properties converge to our baseline model as $\varepsilon \to 0$. Moreover, for a given offer p, the probability that shareholders tender their shares in a responsive equilibrium decreases with ε. As a result, the raider's expected profit also decreases with ε, and following the logic of Proposition 3, he cannot make any profit in the limit.

3.3.2 Tendering cost

As another example, consider a perturbation of the model in which tendering entails a cost $c > 0$ on each tendering shareholder, e.g., the time and effort that the physical action of tendering requires. Non-tendering involves no costs. As long as $c > 0$, even if it is arbitrarily small (or even converges to zero as n gets arbitrarily large), a non-responsive equilibrium does not exist. Indeed, in a non-responsive equilibrium each shareholder expects the freeze-out clause to be triggered with probability one, which means that he would receive p for his share regardless of his tendering decision. Therefore, it is strictly optimal to keep the share and save the tendering cost. By contrast, it can be shown that a unique symmetric responsive equilibrium always exists. Intuitively, given offer p, each shareholder will be indifferent between tendering and not tendering, just as in the baseline model. The key difference is that when $c > 0$, the offer must be attractive enough not only to convince shareholders to avoid free-riding, but also

[16] The same argument works if each non-tendering shareholder receives $p' \in (p, v]$ instead of p.

to compensate them for the tendering cost. The properties of this equilibrium converge to the baseline model as $c \to 0$.

3.3.3 Shareholder inattention

Finally, consider a perturbation of the model in which each shareholder is hit by an attention shock with probability $\rho > 0$. Suppose that a distracted shareholder keeps his share regardless of the offer p. For example, the shareholder cannot respond in a timely fashion to the tender offer due to unexpected events in his personal or professional life. An attentive shareholder is rational as in the baseline model.[17] In this setup, for every $\rho > 0$, even if ρ is arbitrarily small, there is a unique equilibrium. In the unique equilibrium, the decision of attentive shareholders is always responsive to the tender offer. Intuitively, the probability that the freeze-out threshold will not be met due to shareholder inattention can never be ruled out. Therefore, attentive shareholders tender their shares only if the offer is sufficiently high to offset their incentives to free-ride. Moreover, as $\rho \to 0$, attentive shareholders tender their shares with probability one only if the offer converges to v. In other words, this unique equilibrium converges to the responsive equilibrium in our baseline model as $\rho \to 0$.

4 Conditional offers

In general, if $p > 0$ then the raider has strict incentives to renege from his commitment to buy all tendered shares if less than 50% of the outstanding shares have been tendered, thus saving on the losses associated with the purchase of those shares. In this section we consider conditional offers, i.e., the raider buys only if a majority of shares are tendered.

Notice that with conditional offers there always exists a non-responsive equilibrium of the subgame in which the offer fails for sure. This equilibrium, however, is Pareto dominated by all other equilibria. Therefore, and for the same reasons as in the baseline model, we focus attention on the unique responsive equilibrium of the tendering subgame.[18]

[17] Attention shocks are i.i.d and each shareholder's attention shock is his own private information.

[18] As in Gromb (1993), it can be shown that equilibria of the subgame in which the offer fails with probability one are not Trembling-Hand Perfect Nash Equilibrium.

If the offer is conditional, a tendering shareholder receives p for his share if and only if at least $\frac{n}{2}-1$ among all other $n-1$ shareholders also tendered their shares. Therefore, a tendering shareholder expects a value of

$$T(\sigma,p)=p\sum_{j=\frac{n}{2}-1}^{n-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}. \qquad (5)$$

A non-tendering shareholder expects a value of $K(\sigma,p)$, which is given by (1), just as in the baseline model. A shareholder has incentives to tender his share if and only if $T(\sigma,p)\geq K(\sigma,p)$. Therefore, a symmetric responsive equilibrium is characterized by $T(\sigma,p)=K(\sigma,p)$. The next result is analogous to Proposition 1.

Proposition 5 *For any $p\in[0,v]$ there exists a unique symmetric responsive equilibrium of the tendering subgame. In this equilibrium, each shareholder tenders his share with probability $\sigma_C(p)$, which is given by the unique solution of*

$$\frac{p}{v}=\frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=\frac{n}{2}-1}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}. \qquad (6)$$

Moreover, $\sigma_C(p)$ is increases in p and decreases in v and ϕ.

Comparing the right hand side of (6) with the right hand side of (2) reveals that the former is larger than the latter, and hence, shareholders are less likely to tender their shares when the offer is conditional. Intuitively, tendering yields only $T(\sigma,p)<p$ when the offer is conditional, and therefore, the incentives to tender are weaker. Other than that, the dynamic of the two models is quite similar.

Given the equilibrium in the tendering subgame, the raider's expected profit when the offer is conditional is given by

$$\Pi_C(\sigma,p)=\Pi(\sigma,p)+p\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{j}\sigma^{j}(1-\sigma)^{n-j}j, \qquad (7)$$

where $\Pi(\sigma,p)$ is given by (3). Indeed, for a given σ and p, the only difference between conditional and unconditional offers is that in the latter the raider buys the shares even if she does not acquire control, i.e., when the takeover fails. Nevertheless, Lemma 5 in the Appendix shows that $\Pi_C(\sigma_C(p), p) < \Pi(\sigma(p), p)$ for every p. That is, when accounting for shareholders' optimal decision, conditional offers are dominated by unconditional offers. Intuitively, while a conditional offer allows the raider to save by not buying shares above their fundamental value when the takeover fails, it also reduces the incentives of shareholders to tender their shares.

The next proposition extends our main result and shows that freeze-out mergers are ineffective in alleviating the free-rider problem in conditional tender offers. The intuition is similar to Proposition 3.

Proposition 6 *If $B = 0$ and the firm is widely held then the raider's expected profit from a conditional offer is zero, that is,* $\lim_{n\to\infty} \Pi_C(\sigma_C(p), p) = 0$.

4.1 Conditional offers with supermajority control thresholds

In principle, the raider can condition his offer on any number of tendered shares. For example, the raider can require at least αn shares to be tendered, where $\alpha \in (\frac{1}{2}, 1]$. However, an offer with $\alpha \in (\frac{1}{2}, 1]$ requires the raider to forgo a profitable takeover whenever $\frac{n_T}{n} \in [\frac{1}{2}, \alpha)$. In practice, such commitment is not always feasible. Nevertheless, in the Appendix we show that our main result extends to any threshold $\alpha \in [\frac{1}{2}, \phi)$.[19] However, if $\alpha \in [\phi, 1)$ then any offer with a positive premium, even if it is arbitrarily small, is accepted by all shareholders with probability one. That is, any equilibrium of the subgame is non-responsive. Indeed, if $\alpha \in [\phi, 1]$ then shareholders cannot free ride since by design the tender offer cannot succeed without triggering the freeze-out clause. As a result, the raider can extract all the surplus from shareholders.

The power of such "extreme" conditional offers to overcome the free-rider problem was first acknowledged by Bagnoli and Lipman (1988), albeit in a slightly different setup. Bagnoli and Lipman (1988) argued that the raider can extract all the surplus from shareholders if he conditions the tender offer on *all* shares being tendered (i.e., $\alpha = 1$), even without a freeze-out

[19] Propositions 5 and 6 are proved for any $\alpha \in [\frac{1}{2}, \phi)$.

clause. In this case, each shareholder is pivotal for the success of the takeover, and therefore, he cannot free-ride other shareholders.[20] Their argument holds for any number of shareholders.

Nevertheless, conditional offers with supermajority control thresholds have several shortcomings. First, shareholders have to believe that the bidder will walk away from the deal if more than 50% of the shares were tendered but the conditions of the offer were not met. De facto, shareholders may perceive these conditional offers as unconditional, and behave accordingly. Second, these conditional offers are riskier. Indeed, if the offer requires more than 90% of the shares to be tendered (which is the freeze-out threshold in many jurisdictions) then the takeover is more likely to fail: Management and other insiders might not tender if they object the deal, institutional investors may hold different view or information about the standalone value of the company, or some shareholders may simply not pay attention to the tender offer.

Finally, and most importantly, different from Bagnoli and Lipman's (1988) argument, with freeze-out mergers shareholders do not tender to conditional offers with $\alpha \in [\phi, 1)$ because they acknowledge that their vote is pivotal, but rather, because they are *indifferent* between tendering and holding onto their shares when all other shareholders are expected to tender. Therefore, similar to our arguments in Section 3.3, the equilibrium in which all shareholders tender their shares with probability one is not robust to perturbations of the model. For example, suppose tendering entails a small cost as in Subsection 3.3.2. The next result shows that even if the tendering cost is arbitrarily small, conditional offers with $\alpha \in [\phi, 1)$ fail with probability one as the number of shareholders gets arbitrarily large.[21]

Proposition 7 *Consider the perturbed model of Subsection 3.3.2 with a tendering cost* $c > 0$ *where* $\lim_{n\to\infty} nc = \underline{c}$ *and* $\underline{c} > 0$ *can be arbitrarily small. Any conditional offer with* $\alpha \in [\phi, 1)$ *fails with probability one as* $n \to \infty$ *in any symmetric equilibrium.*

[20] Gromb (1993) made a similar argument about unanimity majority requirement. With unanimity, each shareholder is pivotal for the takeover success even if the offer is unconditional.

[21] The fragility of conditional offers with $\alpha \in [\phi, 1)$ can also be demonstrated by perturbing the model with uncertainty about the freeze-out condition. This fragility, however, cannot be obtained when the model is perturbed with shareholder inattention. Intuitively, unlike the other two perturbations, this one does not "break" the indifference of shareholders between tendering and non-tendering. Instead, it introduces the possibility that the freeze-out threshold will not be met. As discussed above, conditional offers with $\alpha \in [\phi, 1)$ are still likely to fail when shareholder inattention is large.

Proposition 7 implies that conditional offers with "extreme" supermajority control thresholds are suboptimal for widely held firms. Indeed, as our main result shows, if $B = 0$ and n is sufficiently large then the raider's expected profit is non-positive no matter how the tender offer is structured. At the same time, and similar to Proposition 4, if $B > 0$, n is sufficiently large, and the tendering cost is arbitrarily small, then the raider can secure a profit of B by making an unconditional tender offer. Therefore, when the firm is widely held and $B > 0$ ($B = 0$), it is strictly (weakly) optimal for the raider to make an unconditional offer or one that is conditioned on less than the freeze-out threshold.

As a final remark, note that the argument behind Proposition 7 also holds when the freeze-out threshold is exactly the simple majority, that is, $\phi = \frac{1}{2}$. In this light, although for a different reason, our proposition that freeze-out mergers do not solve the free-rider problem holds even with a 50% freeze-out threshold.

5 Concluding Remarks

In this paper, we study the role of freeze-out mergers in mitigating free rider problem in takeovers in a finitely many shareholder framework. In contrast to framework that employs infinitesimal shareholders, our framework allows us to characterize an equilibrium in which shareholders' likelihood of tendering depends on the offer price. More specifically, in equilibrium shareholders are more likely to tender following a higher offer price. We show that this equilibrium is the unique symmetric responsive equilibrium and we focus our analysis on this equilibrium.

Raider's ability to freeze out non-tendering shareholders reduces shareholders incentive to free ride. Consequently, the raider can afford to lower the offer price without lowering her success probability if she is allowed to use more aggressive freeze-out clauses. Therefore, if requirements for freeze-out mergers are easy to satisfy, the acquisitions become more profitable for raider.

Despite this finding for a finite shareholder framework, our main result shows that free rider problem precludes value increasing takeovers for widely held firms. In other words, when the number of shareholders gets larger, the expected profit of the raider converges to zero even when the raider is allowed to freeze-out minority shareholders once she has acquired *strictly* more than half of shares. In that sense, Grossman and Hart (1980) and Bagnoli and Lipman (1988) result extends to the freeze-out mergers.

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A Appendix

A.1 Proof of main results

Proof of Proposition 1. Simple algebra shows that $p = K(\sigma, p)$ is equivalent to (2). The left hand side of ("LHS") of (2) is within the interval $[0,1]$ by definition. The expression on the right hand side ("RHS") also takes values within the interval $[0,1]$, since the term on the numerator is (weakly) less than the term on the denominator. Moreover, the RHS can be rewritten as

$$\frac{1}{1+\sum_{i=0}^{\frac{n}{2}-1}\left[\binom{n-1}{i}\left(\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\left(\frac{\sigma}{1-\sigma}\right)^{j-i}\right)^{-1}\right]}. \tag{8}$$

Since, $j \geq \frac{n}{2} > i$ in the summation, it is immediate that the RHS is continuous and increasing in σ, and it converges to 0 and 1 as $\sigma \to 0$ and $\sigma \to 1$, respectively. Therefore, given any p, there exists a unique σ that sets the equality. The comparative statics of σ with respect to p, v, and ϕ, follows directly from the fact that the LHS of (2) is $\frac{p}{v}$ and (8) is an increasing function of σ and ϕ. ■

Proof of Proposition 2. Applying the formulation derived in mathematical digression of Dalkır and Dalkır (2014) to first and second terms of (3), and noting that $(v-p)n = V - P$, we get

$$\begin{aligned}\Pi(\sigma,p) \quad = \quad & V\sigma\sum_{j=\frac{n}{2}-1}^{n\phi-2}\tbinom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} - P\sigma\sum_{j=0}^{n\phi-2}\tbinom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ & +(V-P)\sum_{j=n\phi}^{n}\tbinom{n}{j}\sigma^j(1-\sigma)^{n-j} + B\sum_{j=\frac{n}{2}}^{n}\tbinom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{9}$$

Separating the first term into three terms,

$$\begin{aligned}\Pi(\sigma,p) &= V\sigma\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} - V\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &\quad +V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} - P\sigma\sum_{j=0}^{n\phi-2}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ &\quad +(V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} + B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{10}$$

Equilibrium condition (2) of the tendering subgame can be rewritten as

$$P\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} = V\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}. \tag{11}$$

Substituting the LHS of (11) in the first term of the profit function (10), we have

$$\begin{aligned}\Pi(\sigma,p) &= P\sigma\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} - V\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &\quad +V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} - P\sigma\sum_{j=0}^{n\phi-2}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ &\quad +(V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} + B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{12}$$

Now the first term above cancels out with the fourth except for the final term of its summand:

$$\begin{aligned}\Pi(\sigma,p) &= P\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi} - V\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &\quad +V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} \\ &\quad +(V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} + B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{13}$$

Combining terms, we have the following expression:

$$\begin{aligned}\Pi(\sigma,p) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} - (V-P)\,\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &\quad + (V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} + B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{14}$$

Equation (14) can be written as

$$\begin{aligned}\Pi(\sigma,p) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} \\ &\quad +(V-P)\left[(1-\phi)\binom{n}{n\phi}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)} + \sum_{j=n\phi+1}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}\right] \\ &\quad +B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}.\end{aligned} \tag{15}$$

Using (11) again we get

$$\begin{aligned}&\Pi(\sigma,p) = V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} \\ &+V\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\left[(1-\phi)\binom{n}{n\phi}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)} + \sum_{j=n\phi+1}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}\right] \\ &+B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j},\end{aligned} \tag{16}$$

which is only a function of σ. We assume that ϕ increases with increments greater or equal to $1/n$. Note that the second term of the profit function above is the only term containing ϕ. Of that term, the first factor is decreasing in ϕ since a higher ϕ will add new positive terms to the summand on the denominator. The second factor in brackets can be written as,

$$(1-\phi)\binom{n}{n\phi}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)} + \binom{n}{n\phi+1}\sigma^{n\phi+1}(1-\sigma)^{n(1-\phi)-1} + \sum_{j=n\phi+2}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} \tag{17}$$

for $\phi \leq \frac{n-2}{n}$. Increasing ϕ by $1/n$ yields

$$(1-\phi-\frac{1}{n})\binom{n}{n\phi+1}\sigma^{n\phi+1}(1-\sigma)^{n(1-\phi)-1}+\sum_{j=n\phi+2}^{n}\binom{n}{j}\sigma^{j}(1-\sigma)^{n-j}. \tag{18}$$

The difference of (18) and (17) is negative. When $\phi = \frac{n-1}{n}$, the summands in both expressions vanish, and (18) takes the value zero, so the difference is still negative. Therefore the entire profit function is decreasing in ϕ. ■

The following two technical results are needed for the proofs of Proposition 3 and Proposition 4. The proof of these auxiliary results can be found in Appendix A.2.

Lemma 1 *If* $k \in \{0,1\}$ *and* $\tau \in (0,1]$ *then* $\lim_{n\to\infty}\sum_{j=0}^{n\tau-1}\binom{n-k}{j}\sigma^{j}(1-\sigma)^{n-k-j} = 1$ *for any* $\sigma < \tau$.

Lemma 2 *If* $k \in \{0,1\}$ *and* $\tau \in (0,1]$ *then,* $\lim_{n\to\infty}\sum_{j=0}^{n\tau-1}\binom{n-k}{j}\sigma^{j}(1-\sigma)^{n-k-j} = 0$ *for any* $\sigma > \tau$.

Proof of Proposition 3. In the proof of Proposition 2 we showed that $\Pi(\sigma,p)$ can be expressed as in (14). Suppose $B=0$. Using equation (11) and substituting for P in the third term of (14) we have

$$\begin{aligned}\Pi(\sigma,p) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}-(V-P)\,\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &+ V\left[1-\frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}\right]\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^{j}(1-\sigma)^{n-j}.\end{aligned} \tag{19}$$

Simplifying the argument inside the brackets, we get

$$\begin{aligned}\Pi(\sigma,p) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}-(V-P)\,\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \\ &+ V\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^{j}(1-\sigma)^{n-j}.\end{aligned} \tag{20}$$

First, note that the first two terms of the profit function (20) converge to zero. This immediately follows from Stirling's Approximation.[22] Therefore, it is sufficient to show that

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} = 0. \tag{21}$$

Recall $\phi > \frac{1}{2}$. There are two cases to consider:

1. Suppose $\sigma < \phi$. Applying Lemma 1 for $k = 0$ and $\tau = \phi$, we have $\lim_{n\to\infty} \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} = 0$. The numerator of the first factor in (21) is less than the denominator, so the value of the quotient belongs to $[0, 1]$. Thus $\lim_{n\to\infty} \Pi(\sigma, p) = 0$.

2. Suppose $\sigma \geq \phi > \frac{1}{2}$. Note that

$$\begin{aligned}
\frac{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} &= 1 - \frac{\sum_{j=\frac{n}{2}}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}, \\
&= 1 - \frac{\sum_{j=\frac{n}{2}}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} + \sum_{j=\frac{n}{2}}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}.
\end{aligned}$$

Since $\lim_{n\to\infty} \sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} = 0$ by Lemma 2, when applied to for $k = 1$ and $\tau = \frac{1}{2}$, the entire equation approaches to zero. Since the second factor of (21) corresponds to a probability, it is bounded above by one. Therefore $\lim_{n\to\infty} \Pi(\sigma, p) = 0$. ■

The next technical result is needed for the proof of Proposition 4. Its proof can be found in Appendix A.2.

Lemma 3 *Let $\pi(\sigma) \equiv \Pi(\sigma, p(\sigma))$, where $p(\sigma)$ is the price that induces tendering probability σ as given by (2). Then, $\frac{\partial \pi(\sigma)}{\partial \sigma} > 0 \Leftrightarrow f(\sigma) > \sigma$ where*

$$f(\sigma) = 1 - \frac{V}{B+V} \frac{1}{2} \frac{1}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} \times \left(1 - \frac{1-\phi}{\frac{1}{2}} \frac{\frac{\binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}}{\frac{\binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}} \right) \tag{22}$$

[22] See Bagnoli and Lipman (1988) and Marquez and Yılmaz (2008) for the argument.

Proof of Proposition 4. We proceed in several steps.

There are three cases to consider:

1. First, suppose $\sigma \in (0, \frac{1}{2})$. Applying Lemma 1 when $k = 1$ and $\tau = \phi$ gives

$$\lim_{n\to\infty} \sum_{j=0}^{n\phi-1} \tbinom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} = 1.$$

Therefore,

$$\lim_{n\to\infty} f(\sigma) = 1 - \frac{V}{B+V}\frac{1}{2} + \frac{V}{B+V}\frac{1}{2}\frac{1-\phi}{\frac{1}{2}} \lim_{n\to\infty} \frac{\frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}}{\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}} \geq \frac{1}{2} > \sigma. \tag{23}$$

Overall, if $\sigma \in \left(0, \frac{1}{2}\right)$ then $\lim_{n\to\infty} f(\sigma) > \sigma$.

2. Second, suppose $\sigma \in (\frac{1}{2}, \phi)$. Applying Lemma 1 when $k = 1$ and $\tau = \phi$ gives

$$\lim_{n\to\infty} \sum_{j=0}^{n\phi-1} \tbinom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} = 1.$$

By Stirling's approximation,

$$\lim_{n\to\infty} \tbinom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} = 0.$$

Applying Lemma 2 when $k = 1$ and $\tau = \frac{1}{2}$ gives

$$\lim_{n\to\infty} \sum_{j=0}^{\frac{n}{2}-1} \tbinom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} = 0,$$

and therefore,

$$\lim_{n\to\infty}\sum_{j=0}^{\frac{n}{2}-2}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}=0$$

as well. This implies

$$\lim_{n\to\infty}\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}=\lim_{n\to\infty}\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}+\sum_{j=0}^{\frac{n}{2}-2}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}=1.$$

Overall

$$\lim_{n\to\infty}f(\sigma)=1-\frac{V}{B+V}\frac{1}{2}\left(1-\frac{1-\phi}{\frac{1}{2}}\frac{\lim_{n\to\infty}\frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}}{\lim_{n\to\infty}\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}}\right)=1-\frac{V}{B+V}\frac{1}{2}. \quad (24)$$

We conclude, if $\sigma\in\left(\frac{1}{2},\phi\right)$ then $\lim_{n\to\infty}f(\sigma)=1-\frac{V}{B+V}\frac{1}{2}$.

3. Third, suppose $\sigma\in(\phi,1)$. Applying Lemma 2 when $k=1$ and $\tau\in\left\{\phi,\frac{1}{2}\right\}$ gives

$$\lim_{n\to\infty}\sum_{j=0}^{n\tau-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}=0,$$

and therefore,

$$\lim_{n\to\infty}\sum_{j=0}^{n\tau-2}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}=0$$

as well. This implies

$$\lim_{n\to\infty}\frac{\binom{n-1}{n\tau-1}\sigma^{n\tau-1}(1-\sigma)^{n-n\tau}}{\sum_{j=0}^{n\tau-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}=\lim_{n\to\infty}\frac{\binom{n-1}{n\tau-1}\sigma^{n\tau-1}(1-\sigma)^{n-n\tau}}{\binom{n-1}{n\tau-1}\sigma^{n\tau-1}(1-\sigma)^{n-n\tau}+\sum_{j=0}^{n\tau-2}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}=1.$$

Overall

$$\begin{aligned}
\lim_{n\to\infty} f(\sigma) &= 1-\frac{V}{B+V}\frac{1}{2}\frac{1}{\lim_{n\to\infty}\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} \\
&\quad\times\left(1-\frac{1-\phi}{\frac{1}{2}}\frac{\lim_{n\to\infty}\frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}}{\lim_{n\to\infty}\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}}\right) \\
&= 1-\frac{V}{B+V}\frac{1}{2}\frac{1-\frac{1-\phi}{\frac{1}{2}}}{\lim_{n\to\infty}\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} = -\infty.
\end{aligned}$$

Therefore, if $\sigma \in (\phi, 1)$ then $\lim_{n\to\infty} f(\sigma) < \sigma$.

Next, let $\Phi(n,\sigma)$, $\sigma(n)$, and $p(n)$ be the equilibrium probability of a successful takeover, the tendering probability, and the offer price, respectively, as a function of n. The proofs for parts of the proposition statement are as follows:

i. We prove $\sigma^*_\infty \equiv \min\{\phi, 1-\frac{V}{B+V}\frac{1}{2}\}$. Recall that according to Lemma 3, $\frac{\partial\pi(\sigma)}{\partial\sigma} > 0 \Leftrightarrow f(\sigma) > \sigma$. Part #1 above shows that if $\sigma \in \left(0, \frac{1}{2}\right)$ then $\lim_{n\to\infty} f(\sigma) > \sigma$. Therefore, if $\sigma \in \left(0, \frac{1}{2}\right)$ then $\lim_{n\to\infty} \frac{\partial\pi(\sigma)}{\partial\sigma} > 0$. Part #3 above shows that if $\sigma \in (\phi, 1)$ then $\lim_{n\to\infty} f(\sigma) < \sigma$. Therefore, if $\sigma \in (\phi, 1)$ then $\lim_{n\to\infty} \frac{\partial\pi(\sigma)}{\partial\sigma} < 0$. Notice that setting $\sigma \in \{0, 1\}$ gives the raider a zero profit for any finite n. Therefore, it must be $\sigma^*_\infty \equiv \lim_{n\to\infty}\sigma(n) \in [\frac{1}{2}, \phi]$. In particular, part #2 above shows that if $\sigma \in \left(\frac{1}{2}, \phi\right)$ then $\lim_{n\to\infty} f(\sigma) = 1 - \frac{V}{B+V}\frac{1}{2}$. Therefore, if $\sigma \in (\frac{1}{2}, \phi)$ then $\frac{\partial\pi(\sigma)}{\partial\sigma} > 0$ if and only if $1-\frac{V}{B+V}\frac{1}{2} > \sigma$. Since $\pi(\sigma)$ is continuous in σ and a maximum always exists, it must be $\sigma^*_\infty \equiv \min\{\phi, 1-\frac{V}{B+V}\frac{1}{2}\}$, as required.

iv. We prove that if $B > 0$ then $\lim_{n\to\infty}\Phi(n,\sigma(n)) = 1$. Note that if $B > 0$ then $\sigma^*_\infty > \frac{1}{2}$. Since $\Phi(n,\sigma) = \sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-1-j}$, applying Lemma 2 when $k = 0$, $\tau = \frac{1}{2}$, and $\hat{\sigma} \in \left(\frac{1}{2}, \sigma^*_\infty\right)$ gives $\lim_{n\to\infty}\Phi(n,\hat{\sigma}) = 1$. Since $\Phi(n,\sigma(n)) \geq \Phi(n,\hat{\sigma})$ for n sufficiently large, we have $\lim_{n\to\infty}\Phi(n,\sigma(n)) = 1$ as required.

iii. We prove that if $B > 0$ then $\lim_{n\to\infty} p(n) = v$. Indeed, based on (2), if on the contrary

$\lim_{n\to\infty}\frac{p(n)}{v}<1$, then it must be

$$\lim_{n\to\infty}\frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}}<1\Leftrightarrow\lim_{n\to\infty}\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}>0.$$

However, applying Lemma 2 when $k=1$, $\tau=\frac{1}{2}$, and $\hat{\sigma}\in\left(\frac{1}{2},\sigma_{\infty}^{*}\right)$ gives

$$\lim_{n\to\infty}\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\hat{\sigma}^{j}(1-\hat{\sigma})^{n-1-j}=0.$$

However, since

$$\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}\leq\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\hat{\sigma}^{j}(1-\hat{\sigma})^{n-1-j}$$

for n sufficiently large, we get a contradiction.

ii. We prove that if $B=0$ then $\lim_{n\to\infty}\Phi\left(n,\sigma\left(n\right)\right)<1$ and $\lim_{n\to\infty}p\left(n\right)<v$. Recall that if $B=0$ then $\sigma_{\infty}^{*}=\frac{1}{2}$. If on the contrary $\lim_{n\to\infty}\Phi\left(n,\sigma\left(n\right)\right)=1$ then the argument in part #c above would prove $\lim_{n\to\infty}p\left(n\right)=v$. Based on (2), it would also imply

$$\lim_{n\to\infty}\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}=0.$$

But notice that for any $\sigma\in\left[0,1\right]$,

$$\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}=\sum_{j=\frac{n}{2}}^{n-1}\binom{n-1}{j}(1-\sigma)^{j}\sigma^{n-1-j}.$$

Therefore,

$$\lim_{n\to\infty}\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\left(\sigma\left(n\right)\right)^{j}\left(1-\sigma\left(n\right)\right)^{n-1-j}=\lim_{n\to\infty}\sum_{j=\frac{n}{2}}^{n-1}\binom{n-1}{j}\left(1-\sigma\left(n\right)\right)^{j}\sigma\left(n\right)^{n-1-j}=0,$$

Since $\sigma_\infty^* = \frac{1}{2}$, it must be

$$\lim_{n\to\infty} \sum_{j=\frac{n}{2}}^{n-1} \binom{n-1}{j} (1-\sigma(n))^j \sigma(n)^{n-1-j} = \lim_{n\to\infty} \sum_{j=\frac{n}{2}}^{n-1} \binom{n-1}{j} \sigma(n)^j (1-\sigma(n))^{n-1-j},$$

but this implies

$$\begin{aligned} & \lim_{n\to\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} (\sigma(n))^j (1-\sigma(n))^{n-1-j} \\ = \; & \lim_{n\to\infty} \sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} + \lim_{n\to\infty} \sum_{j=\frac{n}{2}}^{n-1} \binom{n-1}{j} (1-\sigma(n))^j \sigma(n)^{n-1-j} = 0. \end{aligned}$$

Since the term on LHS is exactly one, we get a contradiction. Since $\lim_{n\to\infty} \Phi(n, \sigma(n)) < 1$, the argument in part #c above would prove $\lim_{n\to\infty} p(n) < v$, as required. ■

Proof of Proposition 5. We prove a more general statement: We allow the offer to be conditional on least $n\alpha$ tendered shares, where $\alpha \in [\frac{1}{2}, \phi)$. If $\alpha \in [\frac{1}{2}, \phi)$ then

$$T(\sigma, p) = p \sum_{j=n\alpha-1}^{n-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}. \tag{25}$$

and

$$K(\sigma, p) = v \sum_{j=n\alpha}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} + p \sum_{j=n\phi}^{n-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}. \tag{26}$$

Some algebraic manipulation can show that $T(\sigma, p) = K(\sigma, p)$ if and only if

$$\frac{p}{v} = \frac{\sum_{j=n\alpha}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=n\alpha-1}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}, \tag{27}$$

which is exactly (6) if $\alpha = \frac{1}{2}$. This condition can be rewritten as

$$\frac{p}{v} = \frac{1}{1 + \left(\sum_{j=n\alpha}^{n\phi-1} \frac{\binom{n-1}{j}}{\binom{n-1}{n\alpha-1}} \left[\frac{\sigma}{1-\sigma}\right]^{j-(n\alpha-1)} \right)^{-1}}. \tag{28}$$

Since, $j > n\alpha - 1$ in the summation in the denominator of the RHS of (28), it is immediate that the RHS is continuous and increasing in σ, and it converges to 0 and 1 as $\sigma \to 0$ and $\sigma \to 1$, respectively. Therefore, given any p, there exists a unique σ that solves the equality. The comparative statics of σ with respect to p, v, and ϕ, follows directly from the fact that the LHS of (28) is $\frac{p}{v}$ and its RHS is an increasing function of σ and ϕ. ■

Proof of Proposition 6. We prove a more general statement: We allow the offer to be conditional on least $n\alpha$ tendered shares, where $\alpha \in [\frac{1}{2}, \phi\}$. If $B = 0$ then

$$\Pi_C(\sigma, p) = (v-p) \sum_{j=n\alpha}^{n\phi-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} j + n(v-p) \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j}. \tag{29}$$

Using

$$\begin{aligned} \sum_{j=n\alpha}^{n\phi-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} j &= n \sum_{j=n\alpha}^{n\phi-1} \binom{n-1}{j-1} \sigma^j (1-\sigma)^{n-j} \\ &= n \sum_{i=n\alpha-1}^{n\phi-2} \binom{n-1}{i} \sigma^{i+1} (1-\sigma)^{n-i-1}, \end{aligned}$$

we replace it in the first term of (29) and get

$$\Pi_C(\sigma, p) = (V-P)\sigma \sum_{j=n\alpha-1}^{n\phi-2} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} + (V-P) \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j}, \tag{30}$$

which is the same as

$$\begin{aligned} \Pi_C(\sigma, p) &= V\sigma \sum_{j=n\alpha-1}^{n\phi-2} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} - P\sigma \sum_{j=n\alpha-1}^{n\phi-2} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} \\ &\quad + (V-P) \sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j}. \end{aligned} \tag{31}$$

Decomposing the first term into three and the second term into two we get

$$\Pi_C(\sigma,p) = V\sigma\begin{bmatrix} \sum_{j=n\alpha}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ +\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n-1-(n\alpha-1)} \\ -\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-1-(n\phi-1)} \end{bmatrix} \tag{32}$$
$$-P\sigma\begin{bmatrix} \sum_{j=n\alpha-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ -\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi} \end{bmatrix} + (V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}$$

using (27) from the proof Proposition 5, we cancel the first term in the first brackets with the first term in second brackets and get,

$$\Pi_C(\sigma,p) = V\sigma\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n(1-\alpha)} - (V-P)\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \tag{33}$$
$$+(V-P)\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j},$$

using (27) again to substitute $V-P$ in the third term we get

$$\Pi_C(\sigma,p) = V\sigma\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n(1-\alpha)} - (V-P)\sigma\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n(1-\phi)} \tag{34}$$
$$+V\frac{\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n-n\alpha}}{\sum_{j=n\alpha-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j},$$

As in Proposition 3, the first two terms of $\Pi_C(\sigma,p)$ converge to zero. Also note that $\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}$ is a probability, and hence, bounded by one. Since

$$\begin{aligned}\frac{\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n-n\alpha}}{\sum_{j=n\alpha-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} &= 1-\frac{\sum_{j=n\alpha}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=n\alpha-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} \\ &= 1-\frac{\sum_{j=n\alpha}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n-n\alpha}+\sum_{j=n\alpha}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}.\end{aligned}$$

and $\lim_{n\to\infty}\binom{n-1}{n\alpha-1}\sigma^{n\alpha-1}(1-\sigma)^{n-1-(n\alpha-1)}=0$, we get $\lim_{n\to\infty}\Pi_C(\sigma,p)=0$ as required. ■

Proof of Proposition 7. Suppose the raider makes an offer p conditional on at least $\alpha\in[\phi,1]$

shares being tendered. Consider a symmetric equilibrium of the tendering subgame in which the tendering probability is $\sigma \in [0,1]$. If the shareholder tenders his share, his expected value is $p \sum_{j=\alpha n-1}^{n-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} - c$. If the shareholder holds onto his share, his expected value is $p \sum_{j=\alpha n}^{n-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}$. Therefore, the shareholder tenders his share if and only if $p \geq \frac{c}{\binom{n-1}{\alpha n-1} \sigma^{\alpha n-1} (1-\sigma)^{n-\alpha n}}$, which can be rewritten as

$$P \geq \frac{nc}{\binom{n-1}{\alpha n-1} \sigma^{\alpha n-1} (1-\sigma)^{n-\alpha n}}. \tag{35}$$

Notice that $\lim_{n\to\infty} nc = \underline{c} > 0$. Moreover, by Stirling's approximation $\lim_{n\to\infty} \binom{n-1}{\alpha n-1} \sigma^{\alpha n-1} (1-\sigma)^{n-\alpha n} = 0$. Therefore, the RHS of (35) converges to infinity as $n \to \infty$, and no shareholder tenders with a positive probability for n sufficiently large. ■

A.2 Supplemental and auxiliary results

Proof of Lemma 1. The argument under consideration, $\sum_{j=0}^{n\tau-1} \binom{n-k}{j} \sigma^j (1-\sigma)^{n-k-j}$ is the cumulative distribution function of the binomial distribution with mean $(n-k)\sigma$ and standard deviation $\sqrt{(n-k)\sigma(1-\sigma)}$. Chebyshev's inequality states that for any random variable X with mean μ and standard deviation $\tau > 0$,

$$\Pr\left[\frac{X-\mu}{\tau} \geq q\right] \leq \frac{1}{1+q^2} \tag{36}$$

for any $q > 0$. Choosing $q = \frac{n\tau-(n-k)\sigma}{\sqrt{(n-k)\sigma(1-\sigma)}} = \sqrt{\frac{n-k}{\sigma(1-\sigma)}} \left(\frac{n}{n-k}\tau - \sigma\right)$, we have (36) implying

$$\sum_{j=n\tau}^{n-k} \binom{n-k}{j} \sigma^j (1-\sigma)^{n-k-j} \leq \frac{\sigma(1-\sigma)}{\sigma(1-\sigma) + (n-k)\left(\frac{n}{n-k}\tau - \sigma\right)^2},$$

with requirement $\sqrt{\frac{n-k}{\sigma(1-\sigma)}} \left(\frac{n}{n-k}\tau - \sigma\right) > 0$ which is satisfied by $\sigma < \tau$ for $k \in \{0,1\}$. If $\sigma < \tau$ then the RHS of the inequality above converges to zero as n tends to infinity, and therefore, $\lim_{n\to\infty} \sum_{j=n\tau}^{n-k} \binom{n-k}{j} \sigma^j (1-\sigma)^{n-k-j} = 0$, that is, $\lim_{n\to\infty} \sum_{j=0}^{n\tau-1} \binom{n-k}{j} \sigma^j (1-\sigma)^{n-k-j} = 1$, as required. ■

Proof of Lemma 2. Similar to Lemma 1, we are dealing with binomial distribution, with mean $(n-k)\sigma$. By Hoeffding's inequality, we have $P(\frac{n_T-(n-k)\sigma}{n-k} \geq t) \leq \exp[-2(n-k)t^2]$, for $t \geq 0$ which implies $P(n_T \geq (n-k)(t+\sigma)) \leq \exp[-2(n-1)t^2]$. Following the symmetry of the binomial coefficients, i.e., $\binom{n-k}{j} = \binom{n-k}{n-k-j}$, we have

$$\begin{aligned} P(n_T &\geq (n-k)(t+\sigma)) = P(n-k-n_T \geq (n-k)(t+1-\sigma)) \\ &= P(-n_T \geq (n-k)(t-\sigma)) = P(n_T \leq (n-k)(\sigma-t)) \leq \exp[-2(n-k)t^2]. \end{aligned}$$

Choosing $t = \sigma - \frac{n\tau-1}{n-k}$, we have the inequality

$$\sum_{j=0}^{n\tau-1} \binom{n-k}{j} \sigma^j (1-\sigma)^{n-k-j} \leq \exp\left[-2(n-k)\left(\sigma - \frac{n\tau-1}{n-k}\right)^2\right]$$

with requirement $\sigma - \frac{n\tau-1}{n-k} \geq 0$ which is satisfied by $\sigma > \tau$ and $k \in \{0,1\}$. The LHS of the inequality is the sum under consideration, and the RHS tends to zero as n tends to infinity, as long as $\sigma > \tau$. ■

Proof of Lemma 3. In the proof of Proposition 2 we show that $\Pi(\sigma, p)$ can be expressed as in (14). Combining the second and third term in (14), we have

$$\begin{aligned} \Pi(\sigma,p) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} \qquad (37) \\ &+ (V-P)\left[\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} - \binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\right] + B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}. \end{aligned}$$

Using equation (2) and substituting for P in the second term above we have

$$\begin{aligned} \pi(\sigma) &= V\sigma\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}} \qquad (38) \\ &+ V\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\left[\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} - \binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\right] \\ &+ B\sum_{j=\frac{n}{2}}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}, \end{aligned}$$

Note that for any integer number $1 \leq k \leq n$,

$$\frac{\partial}{\partial \sigma} \sum_{j=0}^{k-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} = -n \binom{n-1}{k-1} \sigma^{k-1} (1-\sigma)^{n-k}. \tag{39}$$

Indeed,

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \sum_{j=0}^{k-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} &= \sum_{j=1}^{k-1} \binom{n}{j} j \sigma^{j-1} (1-\sigma)^{n-j} - \sum_{j=0}^{k-1} \binom{n}{j} (n-j) \sigma^j (1-\sigma)^{n-1-j} \\
&= n \sum_{j=1}^{k-1} \binom{n-1}{j-1} \sigma^{j-1} (1-\sigma)^{n-j} - n \sum_{j=0}^{k-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} \\
&= n \sum_{j=0}^{k-2} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} - n \sum_{j=0}^{k-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j} \\
&= -n \binom{n-1}{k-1} \sigma^{k-1} (1-\sigma)^{n-k}
\end{aligned}$$

Using identity (39), we have

$$\begin{aligned}
&\frac{\partial}{\partial \sigma} \left[\sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - \binom{n-1}{n\phi-1} \sigma^{n\phi} (1-\sigma)^{n-n\phi} \right] \\
&= -\frac{\partial}{\partial \sigma} \sum_{j=0}^{n\phi-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - n \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi-1} [\phi - \sigma] \\
&= n \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi} - n \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi-1} [\phi - \sigma] \\
&= n (1-\phi) \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi-1} > 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \sum_{j=\frac{n}{2}}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} &= -\frac{\partial}{\partial \sigma} \sum_{j=0}^{\frac{n}{2}-1} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} \\
&= n \binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}},
\end{aligned}$$

and

$$\frac{\partial}{\partial\sigma}\left[\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}\right]$$

$$= \frac{\left[\begin{array}{c} -(n-1)\binom{n-2}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{n-1-\frac{n}{2}}\left[\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}\right] \\ +\left[\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}\right](n-1)\binom{n-2}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-1-n\phi}\end{array}\right]}{\left[\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}\right]^{2}}$$

$$= \frac{n-1}{1-\sigma}\frac{\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}\binom{n-2}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}-\binom{n-2}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{n-\frac{n}{2}}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}$$

Since $(n-1)\binom{n-2}{n\phi-1}=n(1-\phi)\binom{n-1}{n\phi-1}$, the derivative above can be written as

$$= \frac{n}{1-\sigma}\frac{\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}(1-\phi)\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}-\frac{1}{2}\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}$$

Overall,

$$\begin{aligned}\frac{\partial\pi(\sigma)}{\partial\sigma} = {} & V\frac{n}{2}\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}-1}(1-2\sigma) \\ & +V\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}n(1-\phi)\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi-1} \\ & +V\frac{n}{1-\sigma}\frac{\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}}(1-\phi)\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}-\frac{1}{2}\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^{j}(1-\sigma)^{n-1-j}} \\ & \times\left[\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^{j}(1-\sigma)^{n-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\right] \\ & +Bn\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}},\end{aligned} \tag{40}$$

and some algebraic manipulation gives

$$\begin{aligned}
\frac{\partial \pi(\sigma)}{\partial \sigma} \frac{1-\sigma}{n} \frac{1}{V} = & \frac{1}{2} \binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}} \\
& \times \left(1 - 2\sigma - \frac{\sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - \binom{n-1}{n\phi-1} \sigma^{n\phi} (1-\sigma)^{n(1-\phi)}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}\right) \\
& + \frac{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} (1-\phi) \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi} \\
& \times \left(1 + \frac{\sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - \binom{n-1}{n\phi-1} \sigma^{n\phi} (1-\sigma)^{n(1-\phi)}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}\right) \\
& + (1-\sigma) \frac{B}{V} \binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}}
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
\frac{\partial \pi(\sigma)}{\partial \sigma} \frac{1-\sigma}{n} \frac{1}{V} = & \left(1 + \frac{B}{V}\right) \binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}+1} \\
& + \left(\begin{array}{c} \frac{\sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} (1-\phi) \binom{n-1}{n\phi-1} \sigma^{n\phi-1} (1-\sigma)^{n-n\phi} \\ -\frac{1}{2} \binom{n-1}{\frac{n}{2}-1} \sigma^{\frac{n}{2}-1} (1-\sigma)^{\frac{n}{2}} \end{array} \right) \\
& \times \left(1 + \frac{\sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - \binom{n-1}{n\phi-1} \sigma^{n\phi} (1-\sigma)^{n(1-\phi)}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}\right).
\end{aligned} \tag{42}$$

Notice that

$$1 + \frac{\sum_{j=n\phi}^{n} \binom{n}{j} \sigma^j (1-\sigma)^{n-j} - \binom{n-1}{n\phi-1} \sigma^{n\phi} (1-\sigma)^{n(1-\phi)}}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}} = \frac{1}{\sum_{j=0}^{n\phi-1} \binom{n-1}{j} \sigma^j (1-\sigma)^{n-1-j}}. \tag{43}$$

Indeed,

$$
\begin{aligned}
& 1+\frac{\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}} \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}\left(\begin{array}{c}\sum_{j=n\phi}^{n}\binom{n}{j}\sigma^j(1-\sigma)^{n-j} \\ +\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\end{array}\right) \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}\left(\begin{array}{c}\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-j}+\sigma^n+\sum_{j=n\phi}^{n-1}\binom{n-1}{j-1}\sigma^j(1-\sigma)^{n-j} \\ +\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\end{array}\right) \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}\left(\begin{array}{c}\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-j}+\sigma^n+\sum_{j=n\phi-1}^{n-2}\binom{n-1}{j}\sigma^{j+1}(1-\sigma)^{n-1-j} \\ +\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\end{array}\right) \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}\left(\begin{array}{c}(1-\sigma)\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ +\sigma\sum_{j=n\phi-1}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ +\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\end{array}\right) \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}\left(\begin{array}{c}\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j} \\ +\sigma\left(\sum_{j=n\phi-1}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}-\sum_{j=n\phi}^{n-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}\right) \\ +\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}-\binom{n-1}{n\phi-1}\sigma^{n\phi}(1-\sigma)^{n(1-\phi)}\end{array}\right) \\
= & \frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}.
\end{aligned}
$$

Using identity (43) on the expression of $\frac{\partial\pi(\sigma)}{\partial\sigma}\frac{1-\sigma}{n}\frac{1}{V}$, we get

$$
\begin{aligned}
\frac{\partial\pi\left(\sigma\right)}{\partial\sigma}\frac{1-\sigma}{n}\frac{1}{V} = & \left(1+\frac{B}{V}\right)\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}\left(1-\sigma\right)^{\frac{n}{2}+1} \\
& +\left(\begin{array}{c}\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\left(1-\phi\right)\binom{n-1}{n\phi-1}\sigma^{n\phi-1}\left(1-\sigma\right)^{n-n\phi} \\ -\frac{1}{2}\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}\left(1-\sigma\right)^{\frac{n}{2}}\end{array}\right) \\
& \times\frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j\left(1-\sigma\right)^{n-1-j}}.
\end{aligned} \tag{44}
$$

Dividing $\frac{\partial\pi(\sigma)}{\partial\sigma}\frac{1-\sigma}{n}\frac{1}{V}$ by $\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}+1}$, we get $\frac{\partial\pi(\sigma)}{\partial\sigma}>0$ if and only if

$$1+\frac{B}{V}+\frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\left(\begin{array}{c}\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}(1-\phi)\frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}+1}}\\ -\frac{\frac{1}{2}\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}+1}}\end{array}\right)>0,$$

which holds if and only if

$$1+\frac{B}{V}+\frac{1}{2(1-\sigma)}\frac{1}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}\left(\frac{1-\phi}{\frac{1}{2}}\frac{\frac{\binom{n-1}{n\phi-1}\sigma^{n\phi-1}(1-\sigma)^{n-n\phi}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}}{\frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}}-1\right)>0.$$

This inequality is equivalent to $f(\sigma)>\sigma$, as required. ■

The next result supplements the discussion that follows Proposition 4.

Lemma 4 *Suppose $B=0$. There exists $\overline{n}>2$ such that for every $n>\overline{n}$ there exists $\overline{\phi}\in(\frac{1}{2},1)$ with the following property: If $\phi\in(\frac{1}{2},\overline{\phi})$ then the optimal offer induces a tendering probability strictly larger than $\frac{1}{2}$.*

Proof. Suppose $B=0$. From Lemma 3 we know that $\frac{\partial\pi(\sigma)}{\partial\sigma}>0\Leftrightarrow f(\sigma)>\sigma$, where $f(\sigma)$ is given by (22). From the proof of Proposition 4 we know that if $\sigma\in(0,\frac{1}{2})$ then $\lim_{n\to\infty}f(\sigma)>\sigma$. Therefore, for a sufficiently large n, $\sigma(n)\geq\frac{1}{2}$. Notice that for a given n, as $\phi\to\frac{1}{2}$ (suppose ϕ changes in steps of $\frac{1}{n}$) then $f\left(\frac{1}{2}\right)\to 1$. Therefore, there exists $\overline{\phi}\in\left(\frac{1}{2},1\right)$ such that if $\phi\in\left(\frac{1}{2},\overline{\phi}\right)$ then $f\left(\frac{1}{2}\right)>\frac{1}{2}$. Under these conditions, if $\sigma\in(0,\frac{1}{2}]$ then $f(\sigma)>\sigma$, and hence, $\frac{\partial\pi(\sigma)}{\partial\sigma}>0$. Therefore, the optimal offer induces $\sigma>\frac{1}{2}$. ■

The next result supplements the analysis in Section 4.

Lemma 5 *If $\alpha=\frac{1}{2}$ then $\Pi_C(\sigma_C(p),p)<\Pi(\sigma(p),p)$ for every $p\in[0,v]$.*

Proof. Using (20) for $\Pi(\sigma,p)$ and (34) for $\Pi_C(\sigma,p)$ when $\alpha=\frac{1}{2}$, the two can be expressed solely as function of σ. Suppose $\Pi(\sigma,p) \leq \Pi_C(\sigma,p)$. If and only if

$$\frac{\sum_{j=0}^{\frac{n}{2}-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} \leq \frac{\binom{n-1}{\frac{n}{2}-1}\sigma^{\frac{n}{2}-1}(1-\sigma)^{\frac{n}{2}}}{\sum_{j=\frac{n}{2}-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} \Leftrightarrow$$

$$\frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=0}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}} \geq \frac{\sum_{j=\frac{n}{2}}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}{\sum_{j=\frac{n}{2}-1}^{n\phi-1}\binom{n-1}{j}\sigma^j(1-\sigma)^{n-1-j}}$$

This is a contradiction since the numerators are the same but the denominator in the RHS is smaller. Therefore $\Pi_C(\sigma_C(p),p) < \Pi(\sigma(p),p)$. ■